Moldova National Olympiad 2008
March 1-2

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Grade 12
Day 1

Problem 1 Consider the equation $x^4 - 4x^3 + 4x^2 + ax + b = 0$, where $a, b \in \mathbb{R}$. Determine the largest value $a+b$ can take, so that the given equation has two distinct positive roots x_1, x_2 so that $x_1 + x_2 = 2x_1x_2$.

Problem 2 Evaluate

$$E = \int_0^{\frac{\pi}{2}} \cos^{1003} x \mathrm{d}x \cdot \int_0^{\frac{\pi}{2}} \cos^{1004} x \mathrm{d}x$$

Problem 3 In the usual coordinate system xOy, line d intersects circles $C_1 : (x+1)^2 + y^2 = 1$ and $C_2 : (x-2)^2 + y^2 = 4$ in the points A, B, C, D (in this order), all having positive Oy coordinates. Given that $A\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ and $m(\angle BOC) = 60^\circ$ find the slope of d.

Problem 4 Define the sequence $(a_p)_{p\geq 0}$ as follows:

$$a_p = \frac{\binom{p}{0}}{2\cdot 4} - \frac{\binom{p}{1}}{3\cdot 5} + \frac{\binom{p}{2}}{4\cdot 6} - \ldots + (-1)^p \cdot \frac{\binom{p}{p}}{(p+2)(p+4)}$$

Compute $\lim_{n\to\infty}(a_0 + a_1 + \ldots + a_n)$.

Day 2

Problem 5 Find the least positive integer n so that the polynomial $P(X) = \sqrt{3}X^{n+1} - X^n - 1$ has at least one root of modulus 1.

Problem 6 For $n \geq 1$, let

$$a_n = \frac{1}{\sqrt{n^2+8n-1}} + \frac{1}{\sqrt{n^2+16n-1}} + \frac{1}{\sqrt{n^2+24n-1}} + \ldots + \frac{1}{\sqrt{9n^2-1}}$$

Find $\lim\limits_{n\to\infty} a_n$.

Problem 7 Vertices B, C of triangle are fixed and $BC = 2$, while A is variable. Denote by H and G the orthocenter and centroid respectively of triangle ABC. Let $F \in (HG)$ so that $HF/FG = 3$. Find the locus of the point A so that $F \in BC$.

Problem 8 Evaluate

$$I = \int_0^{\frac{\pi}{4}} \left(\sin^6 2x + \cos^6 2x\right) \cdot \ln(1 + \tan x)\mathrm{d}x$$